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R-1181

SPHERICAL GEODETIC TRANSFORMATIONS

VOLUME II OF II

CATALOG OF EXPLICIT REPRESENTATIONS IN THE SPATIAL AND FREQUENCY DOMAINS

by

William M. Robertson

September 1978

Prepared for:
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Monitoring Office:
Space and Missile Systems Organization (MNCA)
Norton Air Force Base, California 92409

The Charles Stark Draper Laboratory, Inc.

Cambridge, Massachusetts 02139

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER R-1181	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SPHERICAL GEODETIC TRANSFORMATIONS: Volume II of II; CATALOG OF EXPLICIT REPRESENTATIONS IN THE SPATIAL AND FREQUENCY DOMAINS		5. TYPE OF REPORT & PERIOD COVERED FINAL 9-30-77 thru 9-30-78 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) William M. Robertson		8. CONTRACT OR GRANT NUMBER(s) F04704-78-C-0002
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Charles Stark Draper Laboratory, Inc. Cambridge, Massachusetts 02139		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Services Line Item 0003 Task 3.12
11. CONTROLLING OFFICE NAME AND ADDRESS Defense Mapping Agency Aerospace Center (STT) St. Louis, Missouri 63118		12. REPORT DATE September 1978 13. NUMBER OF PAGES 125
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Space and Missile Systems Organization Air Force Systems Command (MNCA) Norton Air Force Base, California 92409		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE --
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Geodesy Stokes Integral Vening-Meinesz Integral, Spherical Digital Filters,		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (See page iii of document)		

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

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ACKNOWLEDGEMENT

This report was prepared by the Charles Stark Draper Laboratory, Inc. for the Defense Mapping Agency Aerospace Center under Task #3.12 (DMA Geodesy Studies), Services Line Item No. 0003 of Contract F04704-78-C-0002 with the Space and Missile Systems Organization of the Air Force Systems Command.

Support for the performance of the work described in this Report has been provided under the following contracts between U.S. Government agencies and The Charles Stark Draper Laboratory, Inc.:

Task #3.12 (DMA Geodesy Studies), Services Line Item No. 0003 of Contract F04704-78-C-0002 with the Space and Missile Systems Organization of the Air Force Systems Command.

Task #3.12 (DMA Geodesy Studies) of Change Order P00073 of Contract F04701-73-C-0277 with the Space and Missile Systems Organization of the Air Force Systems Command.

Task #1150 (4320) (Gravity Model) of Contract N00030-78-C-0100 with the Strategic Systems Project Office of the U.S. Navy.

Task #4300 (Guidance System Initialization) of Contract N00030-77-C-0070 with the Strategic Systems Project Office of the U.S. Navy.

The author especially wishes to acknowledge the support provided by the U.S. Navy under the last contract listed above. This support came at a very early stage in the author's work, and was the enabling force for the subsequent development and application of the theory as described in this document.

ABSTRACT

This volume is a catalog of the spatial and frequency domain representations of approximately 100 spherical integral transformations, of which about 85 have an explicitly geodetic interpretation.

The following information is provided (when known) for each transformation:

- The input and output quantities of the transformation stated in geodetic nomenclature;
- An explicit analytic double-integral expression for the transformation written in the generally accepted geodetic symbolism;
- An explicit expression for the spectrum of the transformation, or equivalently the transfer function or gain function as it would be called in engineering terminology.
- An explicit Legendre series expansion of the kernel of the transformation, and a closed form expression for the expansion;
- Detailed references to the geodetic and mathematical literature for statements or proofs of the expressions;
- Additional remarks concerning the transformation.

This catalog assembles a rather comprehensive set of results in a consistent format and notation.

PERSONAL ACKNOWLEDGEMENT

The author wishes to express his gratitude and appreciation to Ruth Erickson for her talent and care in the typing and physical preparation of both volumes of this document.

TABLE OF CONTENTS

SECTION 1

EXPLANATION

EXPLANATION

Each page of the catalog contains information about a single spherical integral transformation.

The transformations are ordered primarily according to their input and secondarily according to their output. Both orderings basically follow the sequence: geoid height or anomalous potential (first), gravity anomaly, gravity variation, surface layer density, gravity disturbance, horizontal gravity disturbance or deflection of the vertical, gravity gradient tensor components (last).

The following headings briefly describe the items of information which are provided for each integral transformation (when known). For a full explanation of the theory, the reader should consult Volume I of this document.

Transformation: Under this heading appears the name and the explicit inputs and outputs of the transformation.

The name will be:

- the "classic" name associated with the integral or its kernel, if such a name exists (e.g. Stokes),
- the name of the author who has derived or published results about the integral (e.g. Malkin), or,
- the natural extension of a name used in non-geodetic contexts, especially potential theory, for identical or analogous ideas (e.g. Neumann).

The explicit inputs and outputs are given in generally accepted* symbology and nomenclature. The non-standard symbols and names (e.g. dg and gravity variation) are usually new concepts defined and described in Volume I of this document. The input is assumed to be known every-

*such as Heiskanen-Moritz (1967)

where on the surface of the sphere, and the output is calculated at a single point on (or above) the surface of the sphere by a single evaluation of the explicit form.

Explicit Form: Under this heading appears the analytic representation of the integral transformation in the spatial domain. In other words, the explicit functional dependence of the output (at a point) on the input (over the sphere) is exhibited. Usually the generally accepted* geodetic symbology is utilized.

If no integration limits appear in the explicit form, the double integral is to be evaluated over the entire surface of the unit sphere, in which case the differential surface area element is represented by $d\sigma$.

The factor 4π, which is the total surface area of the unit sphere, is always written in the denominator as a normalizing factor under the differential surface area element. The author has found that this practice simplifies the transformations and spectra by removing "extraneous" constants.

The dependence of the input quantity on the two spherical position parameters ψ and α is not usually indicated. For example, Stokes' Integral contains only the gravity anomaly symbol Δg in the integrand rather than the symbol $\Delta g(\psi,\alpha)$. This notational abbreviation, which is often used in the literature, should not cause any confusion since it is assumed that the input quantity to the transformation is known everywhere on the sphere.

Spectrum: Under this heading appears the analytic representation of the integral transformation in the spherical frequency domain. In mathematical terminology this is the spectrum of the transformation, while in electrical engineering it is called the transfer function or gain function. As explained in Volume I of this document, the spectrum is discrete, having one value λ_n for each spherical harmonic degree n, where $n=0,1,2,\ldots,\infty$. The spectrum is the Legendre Transform of the kernel of the integral transformation. The exact mathematical relationships are summarized on the first page of Section 1 of this volume. In many cases, especially when no references are listed, the spectrum has been derived from a "flow diagram" of spectra. Flow diagrams are described in Section 2 of Volume I of this document.

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*such as Heiskanen-Moritz (1967)

Kernel: Under this heading appears the spherical harmonic "decomposition" or "expansion" of the kernel. This is equivalent to the Inverse Legendre Transform of the spectrum. Again, the exact mathematical relationships are summarized on the first page of Section 1 of this volume.

References and Remarks: Self-explanatory.

SECTION 2

GENERAL SPHERICAL TRANSFORMATION RELATIONSHIPS

AND

CERTAIN SPECIAL GEODETIC TRANSFORMATIONS

GENERAL RELATIONSHIPS

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: $f_{OUT}(0,0)$

EXPLICIT FORM:

$$\underline{f}_{OUT}(0,0) = \iint \underline{\mathscr{K}}(\psi,\alpha)\, f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi}$$

where $d\sigma = \sin\psi\, d\psi\, d\alpha$ = element of surface area

SPECTRUM:

$$\underline{\lambda}_n^m = \varepsilon_m \sqrt{\frac{(n-m)!}{(n+m)!}} \iint \underline{\mathscr{K}}(\psi,\alpha)\, P_n^m(\cos\psi) \begin{Bmatrix} \cos m\alpha \\ \\ \sin m\alpha \end{Bmatrix} \frac{d\sigma}{4\pi}$$

KERNEL:

$$\underline{\mathscr{K}}(\psi,\alpha) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \underline{\lambda}_n^m (2n+1) \sqrt{\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\psi) \begin{Bmatrix} \cos m\alpha \\ \\ \sin m\alpha \end{Bmatrix}$$

REFERENCES:

REMARKS:

- ε_m = Neumann Factor = $(2 - \delta_{m0}) = \begin{cases} 1 \text{ when } m = 0 \\ \\ 2 \text{ when } m \neq 0 \end{cases}$

- The underline denotes a two-dimensional (vector) function.

SPECIFIC RELATIONSHIPS
ZERO-TH ORDER KERNELS

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: $f_{OUT}(0,0)$

EXPLICIT FORM:

$$f_{OUT}(0,0) = \iint K(\cos\psi)\, f(\psi,\alpha)\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \int_{-1}^{+1} K(\cos\psi)\, P_n\,(\cos\psi)\frac{d(\cos\psi)}{2}$$

KERNEL:

$$K(\cos\psi) = \sum_{n=0}^{\infty} \lambda_n(2n+1)\, P_n\,(\cos\psi)$$

REFERENCES:

REMARKS:

- Stokes' kernel is the most common example of a zero-th order kernel.
- The zero-th order kernels are isotropic, or independent of azimuth α.

SPECIFIC RELATIONSHIPS
FIRST ORDER KERNELS

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: $\underline{f}_{OUT}(0,0)$

EXPLICTI FORM:

$$\underline{f}_{OUT}(0,0) = \iint K(\cos\psi)\,\underline{A}(\alpha)\, f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\underline{\lambda}_n^1 = \frac{2}{\sqrt{n(n+1)}} \int_{-1}^{+1} K(\cos\psi)\, P_n^1(\cos\psi)\frac{d(\cos\psi)}{2}\int_0^{2\pi} \underline{A}(\alpha)\begin{Bmatrix}\cos\alpha \\ \\ \sin\alpha\end{Bmatrix}\frac{d\sigma}{2\pi}$$

KERNEL:

$$K(\cos\psi)\,\underline{A}(\alpha) = \sum_{n=1}^{\infty} \underline{\lambda}_n^1 \frac{(2n+1)}{\sqrt{n(n+1)}} P_n^1(\cos\psi)\begin{Bmatrix}\cos\alpha \\ \\ \sin\alpha\end{Bmatrix}$$

REFERENCES:

REMARKS:

It is assumed that the general kernel $\underline{\mathcal{K}}(\psi,\alpha)$ is separable into radial and azimuthal parts:

$$\underline{\mathcal{K}}(\psi,\alpha) = K(\cos\psi)\,\underline{A}(\alpha)$$

TRANSFORMATION: UPWARD CONTINUATION, POISSON

Input: $V(R,\psi,\alpha)$ = harmonic function on sphere of radius R

Output: $V(r,0,0)$ = harmonic function at radius $r > R$

EXPLICIT FORM:

$$V(r,0,0) = \iint \frac{R(r^2 - R^2)}{(r^2 - 2Rr\cos\psi + R^2)^{3/2}} V(R,\psi,\alpha) \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \left(\frac{R}{r}\right)^{n+1}$$

KERNEL:

$$\frac{R(r^2 - R^2)}{(r^2 - 2Rr\cos\psi + R^2)^{3/2}} = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} (2n+1)\, P_n\, (\cos\psi)$$

REFERENCES:

Heiskanen-Moritz, page 35, equation 1-89.

REMARKS:

TRANSFORMATION: IDENTITY, DELTA FUNCTION

Input: $V(R,\psi,\alpha)$ = harmonic function on sphere of radius R

Output: $V(R,0,0)$ = harmonic function at a point at radius R

EXPLICIT FORM:

$$V(R,0,0) = \iint \delta_{2D}(\psi)\ V(R,\psi,\alpha)\ \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = 1$$

KERNEL:

$$\delta_{2D}(\psi) = \lim_{r \to R} \frac{R(r^2 - R^2)}{(r^2 - 2rR\cos\psi + R^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1)\ P_n\ (\cos\psi)$$

REFERENCES:

REMARKS:

- $\delta_{2D}(\psi) = 1 + 2\ \ell n(\sin\frac{\psi}{2})\ \nabla^2_{SURFACE}$

  where $\nabla^2_{SURFACE}$ is the surface Laplacian operator.

- The symbol $\delta_{2D}(\psi)$ is used to emphasize that this "function" is not equal to the one-dimensional delta function.

SECTION 3

SPHERICAL GEODETIC TRANSFORMATIONS

WITH

GEOID HEIGHT AS INPUT

TRANSFORMATION: INVERSE STOKES, MOLODENSKII

Input: $N = T/G$ = geoid height

Output: Δg = gravity anomaly

EXPLICIT FORM:

$$\Delta g_P = \frac{G}{R}\left[-N_P - 2\iint \frac{(N - N_P)}{(2\sin\frac{\psi}{2})^3}\,\frac{d\sigma}{4\pi}\right]$$

SPECTRUM:

$$\lambda_n = \frac{G}{R}\begin{cases} -1 & \text{for } n = 0 \\ 0 & \text{for } n = 1 \\ n-1 & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$K(\cos\psi) = \frac{G}{R}\left[-\delta_{2D}(\psi) + M(\psi)\right] = \frac{G}{R}\sum_{n=2}^{\infty}(n-1)(2n+1)\,P_n\,(\cos\psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 243, equation 697.
Molodenskii, page 50, equation III.2.4.

REMARKS:

- $M(\psi)$ is the kernel for the transformation from N to dg.

- Equation VIII.4.16 on page 214 of Molodenskii is incorrect. It should read:

$$\frac{1}{r^3} = -\sum n(2n+1)\,P_n\,(\cos\psi)$$

TRANSFORMATION: TRUNCATED MOLODENSKII

Input: $N = T/G$ = geoid height

Output: $\Delta g_{(\psi_0)}$ = gravity anomaly computed only from data within a spherical cap of radius ψ_0.

EXPLICIT FORM:

$$\Delta g_{P(\psi_0)} = \frac{G}{R}\left[-N_P - 2\int_0^{\psi_0}\int_0^{2\pi} \frac{(N - N_P)}{(2\sin\frac{\psi}{2})^3} \frac{\sin\psi\, d\alpha\, d\psi}{4\pi}\right]$$

SPECTRUM:

$$\lambda_n = \frac{G}{R}\begin{cases} 0 - \dfrac{R_n(\psi_0)}{2} & \text{for } n = 0,1 \\ (n-1) - \dfrac{R_n(\psi_0)}{2} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

REFERENCES:

Buglia, NASA TMX-72798

REMARKS:

The $R_n(\psi_0)$ function here equals Buglia's R_n function divided by eight. This normalization makes the $R_n(\psi_0)$ functions analogous to Molodenskii's $Q_n(\psi_0)$ functions.

TRANSFORMATION: RESIDUAL MOLODENSKII

Input: $N = T/G$ = geoid height

Output: $\delta(\Delta g)_{\psi_0}$ = residual gravity anomaly due to data outside spherical cap of radius ψ_0.

EXPLICIT FORM:

$$\delta(\Delta g)_{\psi_0} = \frac{G}{R}\left[-N_p - 2\int_{\psi_0}^{\pi}\int_{0}^{2\pi}\frac{(N - N_p)}{(2\sin\frac{\psi}{2})^3}\ \frac{\sin\psi\ d\alpha\ d\psi}{4\pi}\right]$$

SPECTRUM:

$$\lambda_n = \frac{G}{R}\,\frac{R_n(\psi_0)}{2}$$

KERNEL:

$$K(\cos\psi) = \frac{G}{R}\sum_{n=0}^{\infty}\frac{R_n(\psi_0)}{2}\,(2n+1)\,P_n\,(\cos\psi)$$

REFERENCES:

Buglia, NASA TMX-72798

Lelgemann, OSU Report #239, pages 40-47.

REMARKS:

The $R_n(\psi_0)$ function here equals Buglia's R_n function divided by eight. This normalization makes these functions analogous to Molodenskii's $Q_n(\psi_0)$ functions.

TRANSFORMATION: MOLODENSKII ANALOG FOR GRAVITY VARIATION

Input: $N = T/G$ = geoid height

Output: dg = gravity variation

EXPLICIT FORM:

$$dg_p = \frac{G}{R}\iint \frac{-2(N - N_p)}{(2 \sin \frac{\psi}{2})^3} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{G}{R} n$$

KERNEL:

$$K(\cos \psi) = \frac{G}{R} M(\psi) = \frac{G}{R} \sum_{n=0}^{\infty} n(2n+1) P_n (\cos \psi)$$

formally

REFERENCES:

Heiskanen-Moritz; page 39, equation 1-102; also pages 307-310.
Meissl, OSU Report 151, page 22, equation 3-11.

REMARKS:

The kernel for this transformation is called $M(\psi)$ by definition.

TRANSFORMATION: MOLODENSKII ANALOG FOR SURFACE LAYER DENSITY

Input: $N = T/G$ = geoid height

Output: μ = surface layer density

EXPLICIT FORM:

$$\mu_p = \frac{G}{R}\left[+\frac{N_p}{2} - 2\iint \frac{(N - N_p)}{(2 \sin \frac{\psi}{2})^3} \frac{d\sigma}{2\pi}\right]$$

SPECTRUM:

$$\lambda_n = \frac{G}{R}\left(\frac{2n+1}{2}\right) = \frac{G}{R}\left(n + \frac{1}{2}\right)$$

KERNEL:

$$K(\cos\psi) = \frac{G}{R}\left[\frac{1}{2}\,\delta_{2d}(\psi) + M(\psi)\right] = \frac{G}{R}\sum_{n=0}^{\infty} \frac{(2n+1)^2}{2} P_n(\cos\psi)$$

formally

REFERENCES:

Heiskanen-Moritz, pages 236-238.

REMARKS:

TRANSFORMATION: INVERSE NEUMANN

Input: $N = T/G$ = geoid height

Output: δg = gravity disturbance

EXPLICIT FORM:

$$\delta g_P = \frac{G}{R}\left[+N_P - 2\iint \frac{(N - N_P)}{(2\sin\frac{\psi}{2})^3}\,\frac{d\sigma}{4\pi}\right]$$

SPECTRUM:

$$\lambda_n = \frac{G}{R}\,(n+1)$$

KERNEL:

$$K(\cos\psi) = \frac{G}{R}\left[\delta_{2D} + M(\psi)\right] = \frac{G}{R}\sum_{n=0}^{\infty}(n+1)(2n+1)\,P_n(\cos\psi)$$

formally

REFERENCES:

Heiskanen-Moritz, pages 37-39, especially equation 1-100.

REMARKS:

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $d_2 g$ = generalized gravity anomaly of order 2.

EXPLICIT FORM:

$$(d_2 g)_P = \frac{G}{R}\left[2\,N_P - 2\iint \frac{(N - N_P)}{(2\sin\frac{\psi}{2})^3}\,\frac{d\sigma}{4\pi}\right]$$

SPECTRUM:

$$\lambda_n = \frac{G}{R}\,(n+2)$$

KERNEL:

$$K(\cos\psi) = \frac{G}{R}\left[2\,\delta_{2D}(\psi) + M(\psi)\right] = \frac{G}{R}\sum_{n=0}^{\infty}(n+2)(2n+1)\,P_n(\cos\psi)$$

formally

REFERENCES:

Hagiwara, Bull.Geod. 50 (1972), 131-135.
(For the second remark only).

REMARKS:

- This is the same transformation as the one transforming the quantity $(\delta g/G)$ into the vertical gradient of the gravity disturbance.
- Based on the reference, it is conjectured that the kernel $K(\cos\psi) = 2 + 9\cos\psi + 2/(2\sin\frac{\psi}{2})^3$.

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $G\varepsilon$ = magnitude of horizontal disturbance

EXPLICIT FORM:

$$G\varepsilon = -\frac{G}{R}\left|\nabla_{SURFACE}\, N\right|$$

where $\nabla_{SURFACE}$ is the surface gradient operator

SPECTRUM:

$$\lambda_n^1 = \frac{G}{R}\sqrt{n(n+1)}$$

KERNEL:

$$-\frac{G}{R}\left|\nabla_{SURFACE}\, N\right| = \frac{G}{R}\sum_{n=0}^{\infty}\frac{(2n+1)}{\sqrt{n(n+1)}}P_n^1(\cos\psi) = \frac{G}{R}\sum_{n=0}^{\infty}(2n+1)\,P_n^1(\cos\psi)$$

REFERENCES:

Heiskanen-Moritz, page 262, equation 7-37.
Meissl, OSU Report 151; page 13, equation 2-19; page 46, Table 1.

REMARKS:

The symbol ε represents the total deflection of the vertical in radians.

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $G\vec{\varepsilon}$ = horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\vec{\varepsilon} = -\frac{G}{R}\,\nabla_{SURFACE}\,N$$

where $\nabla_{SURFACE}$ is the surface gradient operator

SPECTRUM:

$$\lambda_n^1 = \frac{G}{R}\sqrt{n(n+1)}$$

KERNEL:

$$-\frac{G}{R}\,\nabla_{SURFACE} = \frac{G}{R}\sum_{n=0}^{\infty}\sqrt{n(n+1)}\;\frac{(2n+1)}{\sqrt{n(n+1)}}\;P_n^1(\cos\psi)\begin{Bmatrix}\cos\alpha\\ \sin\alpha\end{Bmatrix}$$

REFERENCES:

Heiskanen-Moritz, page 112, equation 2-203
Meissl, OSU Report 151, page 46, Table 1.

REMARKS:

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $\frac{\partial}{\partial r} \Delta g$ = vertical gradient of gravity anomaly

EXPLICIT FORM:

$$\frac{\partial}{\partial r} \Delta g = \frac{G}{R^2} \left[2N + \nabla^2_{\text{SURFACE}} N \right]$$

where $\nabla^2_{\text{SURFACE}}$ is the surface Laplacian operator.

SPECTRUM:

$$\lambda_n = \frac{G}{R^2} [2 - n(n+1)] = \frac{-G}{R^2} (n-1)(n+2)$$

KERNEL:

REFERENCES:

Meissl, OSU Report 151; page 44, equation 5-21; page 46, Table I.
Heiskanen-Moritz, page 116, equation 219.

REMARKS:

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $\frac{\partial \mu}{\partial r}$ = vertical gradient of surface layer density

EXPLICIT FORM:

$$\frac{\partial \mu}{\partial r} = -\frac{G}{R^2}\ \frac{N}{4} - \frac{3}{2}\,\frac{\mu}{R} + \frac{G}{R^2}\,\nabla^2_{SURFACE}\ N$$

SPECTRUM:

$$\lambda_n = \frac{G}{R^2}\left[-\frac{1}{4} - \frac{3}{2}\,(n+\tfrac{1}{2}) - n(n+1)\right] = -\frac{G}{R^2}\,(n+\tfrac{1}{2})(n+2)$$

KERNEL:

REFERENCES:

REMARKS:

The explicit form has been derived from the spectrum.

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $\frac{\partial}{\partial r} dg$ = vertical gradient of gravity variation

EXPLICIT FORM:

$$\frac{\partial}{\partial r} dg = -\frac{dg}{R} + \frac{G}{R^2} \nabla^2_{SURFACE} N$$

SPECTRUM:

$$\lambda_n = \frac{G}{R^2} [-n - n(n+1)] = -\frac{G}{R^2} n(n+2)$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $\frac{\partial}{\partial r}\delta g$ = vertical gradient of gravity disturbance

EXPLICIT FORM:

$$\frac{\partial\ \delta g}{\partial r} = -\frac{2\ \delta g}{R} + \frac{G}{R^2}\nabla^2_{SURFACE} N$$

SPECTRUM:

$$\lambda_n = \frac{G}{R^2}\left[-2(n+1) - n(n+1)\right] = -\frac{G}{R^2}(n+1)(n+2)$$

KERNEL:

REFERENCES:

REMARKS:

The explicit form has been derived from the spectrum.

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $\frac{\partial\, d_2 g}{\partial r}$ = vertical gradient of generalized gravity anomaly of order two

EXPLICIT FORM:

SPECTRUM:

$$\lambda_n = -\frac{G}{R^2}(n+2)^2$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $\frac{\partial \, G\varepsilon}{\partial r}$ = vertical gradient of magnitude of horizontal disturbance

EXPLICIT FORM:

SPECTRUM:

$$\lambda_n^1 = -\frac{G}{R^2} \sqrt{n(n+1)} \, (n+2)$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: $N = T/G$ = geoid height

Output: $\frac{\partial G\vec{\varepsilon}}{\partial r} = (T_{xz}, T_{yz})$ = vertical shear gravity gradients

EXPLICIT FORM:

$$\left.\begin{matrix} T_{xz} \\ T_{yz} \end{matrix}\right\} = -G \frac{\partial}{\partial h} (\nabla_{SURFACE} N)$$

SPECTRUM:

$$\lambda_n^1 = -\frac{G}{R^2} \sqrt{n(n+1)}\ (n+2)$$

KERNEL:

REFERENCES:

REMARKS:

SECTION 4

SPHERICAL GEODETIC TRANSFORMATIONS WITH GRAVITY ANOMALIES AS INPUT

TRANSFORMATION: STOKES

Input: Δg = gravity anomaly

Output: $N = T/G$ = geoid height

EXPLICIT FORM:

$$N = \frac{T}{G} = \frac{R}{G} \iint S(\psi)\ \Delta g\ \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{R}{G} \begin{cases} 0 & \text{for } n = 0,1 \\ \frac{1}{n-1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$\frac{R}{G}\, S(\psi) = \frac{R}{G} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n\ (\cos\psi)$$

REFERENCES:

Heiskanen-Moritz, pages 92-98.
Meissl, OSU Report 151, page 22, equation 3-10.

REMARKS:

$$S(\psi) = \frac{1}{\sin\frac{\psi}{2}} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi\ \ell n\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)$$

TRANSFORMATION: TRUNCATED STOKES

Input: Δg = gravity anomaly

Output: $N_{(\psi_0)}$ = geoid height calculated only from data within a spherical cap of radius ψ_0.

EXPLICIT FORM:

$$N_{(\psi_0)} = \frac{R}{G} \int_0^{\psi_0} \int_0^{2\pi} S(\psi)\ \Delta g\ \frac{\sin\psi\ d\alpha\ d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{R}{G} \begin{cases} 0 - \dfrac{Q_n(\psi_0)}{2} & \text{for } n = 0,1 \\ \dfrac{1}{n-1} - \dfrac{Q_n(\psi_0)}{2} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$K(\cos\psi) = \frac{R}{G}\ [S(\psi) - \bar{S}(\psi)] = \sum_{n=0}^{\infty} \lambda_n (2n+1)\ P_n\ (\cos\psi)$$

REFERENCES:

Heiskanen-Moritz, pages 259-263.
Molodenskii, pages 147-150.

REMARKS:

The $Q_n(\psi_0)$ functions are the Molodenskii functions defined in the references.

TRANSFORMATION: RESIDUAL STOKES

Input: Δg = gravity anomaly

Output: $\delta N_{(\psi_0)}$ = residual geoid height due to data outside spherical cap of radius ψ_0.

EXPLICIT FORM:

$$\delta N_{(\psi_0)} = \frac{R}{G} \int_{\psi_0}^{\pi} \int_{0}^{2\pi} S(\psi)\ \Delta g\ \frac{\sin\psi\ d\alpha\ d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{R}{G} \frac{Q_n(\psi_0)}{2}$$

KERNEL:

$$K(\cos\psi) = \frac{R}{G}\,\bar{S}(\psi) = \frac{R}{G} \sum_{n=0}^{\infty} \frac{Q_n(\psi_0)}{2}\,(2n+1)\ P_n\ (\cos\psi)$$

REFERENCES:

Heiskanen-Moritz, pages 259-263.

Molodenskii, pages 147-150.

REMARKS:

The $Q_n(\psi_0)$ functions are the well-known Molodenskii functions defined in the references. From the above it is seen that they have a very elegant interpretation in the spectral theory of geodetic transformations.

TRANSFORMATION: HELMERT

Input: Δg = gravity anomaly

Output: μ = surface layer density

EXPLICIT FORM:

$$\mu = \Delta g + \frac{3}{2}\frac{G}{R}N = \iint \left[\delta_{2D}(\psi) + \frac{3}{2}S(\psi)\right]\Delta g \, \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 1 & \text{for } n = 0,1 \\ \dfrac{2n+1}{2(n-1)} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

REFERENCES:

Heiskanen-Moritz, pages 236-238.

Pick-Picha-Vyskocil, page 240, equation 685.

REMARKS:

TRANSFORMATION:

Input: Δg = gravity anomaly

Output: dg = gravity variation

EXPLICIT FORM:

$$dg = \Delta g + \frac{G}{R} N = \iint \left[\delta_{2D}(\psi) + S(\psi) \right] \Delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 1 & \text{for } n = 0,1 \\ \frac{n}{n-1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: Δg = gravity anomaly

Output: δg = gravity disturbance

EXPLICIT FORM:

$$\delta g \approx \Delta g + 2\frac{G}{R}N = \iint \left[\delta_{2D}(\psi) + 2S(\psi)\right]\Delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 1 & \text{for } n = 0,1 \\ \frac{n+1}{n-1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: Δg_R = gravity anomaly on sphere of radius R

Output: δg_r = gravity disturbance at radius r.

EXPLICIT FORM:

$$\delta g_r = -\iint R \frac{\partial S(r,\psi)}{\partial r} \Delta g_R \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0,1 \\ \left(\frac{R}{r}\right)^{n+2} \left(\frac{n+1}{n-1}\right) & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$K(r,\psi) = \frac{-R\partial S(r,\psi)}{\partial r} = \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+2} \left(\frac{n+1}{n-1}\right) (2n+1) P_n (\cos\psi)$$

REFERENCES:

Heiskanen-Moritz, page 234, equation 6-39a.

REMARKS:

Heiskanen-Moritz uses the symbol δ_r to represent the negative of the gravity disturbance δg_r at the radius r. See Heiskanen-Moritz, page 236, equation 6-50.

TRANSFORMATION:

Input Δg = gravity anomaly

Output: $d_2 g$ = generalized gravity anomaly of order two

EXPLICIT FORM:

SPECTRUM:

$$\lambda_n = \begin{cases} ? & \text{for } n = 0,1 \\ \dfrac{n+2}{n-1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION: TOTAL VENING-MEINESZ

Input: Δg = gravity anomaly

Output: $G\varepsilon$ = magnitude of horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\varepsilon = \iint \frac{\partial S(\psi)}{\partial \psi} \Delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \begin{cases} 0 & \text{for } n = 1 \\ \dfrac{\sqrt{n(n+1)}}{n-1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$\frac{\partial S(\psi)}{\partial \psi} = \sum_{n=2}^{\infty} \frac{\sqrt{n(n+1)}}{n-1} \frac{(2n+1)}{\sqrt{n(n+1)}} P_n^1(\cos\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n^1(\cos\psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 273, equation 859
Meissl, OSU Report 151, page 46, Table I.

REMARKS:

Pick-Picha-Vyskocil uses the symbol $P_n^1(\cos\psi)$ for the function which Heiskanen-Moritz and this treatise denote by $P_{n1}(\cos\psi)$ and which is the negative of our $P_n^1(\cos\psi)$. Hence Pick-Picha-Vyskocil has a negative sign in equation 859.

TRANSFORMATION: VENING-MEINESZ

Input: Δg = gravity anomaly

Output: $G\vec{\varepsilon}$ = horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\vec{\varepsilon} = \begin{Bmatrix} G\xi \\ G\eta \end{Bmatrix} = \iint \frac{\partial S(\psi)}{\partial \psi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} \Delta g \, \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \begin{cases} 0 & \text{for } n = 1 \\ \dfrac{\sqrt{n(n+1)}}{n-1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$\frac{\partial S(\psi)}{\partial \psi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} = \sum_{n=2}^{\infty} \frac{\sqrt{n(n+1)}}{n-1} \, \frac{2n+1}{\sqrt{n(n+1)}} \, P_n^1(\cos\psi) \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix}$$

$$= \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \, P_n^1(\cos\psi) \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix}$$

REFERENCES:

Heiskanen-Moritz, pages 111-114.
Pick-Picha-Vyskocil, page 273, equation 859.
Meissl, OSU Report 151, page 46, Table I.

REMARKS:

Pick-Picha-Vyskocil uses the symbol $P_n^1(\cos\psi)$ for the function which Heiskanen-Moritz and this treatise denote by $P_{n1}(\cos\psi)$ and which is the negative of our $P_n^1(\cos\psi)$. Hence Pick-Picha-Vyskocil has a negative sign in equation 859.

TRANSFORMATION:

Input: Δg = gravity anomaly

Output: $\frac{\partial \Delta g}{\partial r}$ = vertical gradient of gravity anomaly

EXPLICIT FORM:

$$\frac{\partial \Delta g}{\partial r} = -\frac{2}{R} \Delta g_P + \frac{2}{R} \iint \frac{(\Delta g - \Delta g_P)}{(2 \sin \frac{\psi}{2})^3} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = -\frac{1}{R} (n+2)$$

KERNEL:

See Pick-Picha-Vyskocil, page 313, equation 1042.

REFERENCES:

Heiskanen-Moritz, page 115, equation 2-217.
Meissl, OSU report 151, page 46, Table I.
Pick-Picha-Vyskocil, page 313 ff, especially equation 1043

REMARKS:

This is the same transformation as the one transforming N (geoid height) into d_2g (the generalized gravity anomaly of order two).

TRANSFORMATION:

Input: Δg = gravity anomaly

Output: $\frac{\partial\, \delta g}{\partial r} = -T_{zz}$ = vertical gradient of gravity disturbance

EXPLICIT FORM:

$$\frac{\partial\, \delta g}{\partial r} = -T_{zz} = -\iint R \, \frac{\partial^2 S(r,\psi)}{\partial r^2} \, \Delta g \, \frac{d\sigma}{4\pi}$$

where the partial derivative is evaluated at $r = R$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0,1 \\ -\frac{1}{R} \frac{(n+2)(n+1)}{(n-1)} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$-R \left. \frac{\partial^2 S(r,\psi)}{\partial r^2} \right|_{r=R} = \sum_{n=2}^{\infty} - \frac{(n+2)(n+1)}{R(n-1)} (2n+1) \, P_n \, (\cos \psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 313, equation 1038.

REMARKS

An analytic expression for $\frac{\partial^2 S(r,\psi)}{\partial r^2}$ is given in Reed, OSU Report 201, page 71.

TRANSFORMATION:

Input: Δg = gravity anomaly

Output: $\frac{\partial\ G\vec{\varepsilon}}{\partial r} = (T_{xz},\ T_{yz})$ = vertical shear gravity gradients

EXPLICIT FORM:

$$\left.\begin{matrix} T_{xz} \\ \\ T_{yz} \end{matrix}\right\} = \iint \frac{\partial^2 S(r,\psi)}{\partial\psi\ \partial r} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} \Delta g\ \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = -\frac{1}{R} \begin{cases} 0 & \text{for } n = 1 \\ \\ \frac{n+2}{n-1}\sqrt{n(n+1)} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$\left.\frac{\partial^2 S(r,\psi)}{\partial\psi\ \partial r}\right|_{r=R} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} = \sum_{n=2}^{\infty} -\frac{(n+2)}{R(n-1)}\ (2n+1)\ P_n^1\ (\cos\psi) \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix}$$

REFERENCES:

REMARKS:

An analytic expression for $\frac{\partial^2 S(r,\psi)}{\partial\psi\ \partial r}$ is given by Reed, OSU Report 201, page 72, equation 5.36.

TRANSFORMATION:

Input: Δg = gravity anomaly

Output: $T_{xx} - T_{yy}$ = difference of horizontal stress gradients

T_{xy} = horizontal shear gradient

EXPLICIT FORM:

$$\left.\begin{matrix} T_{xx} - T_{yy} \\ \\ 2T_{xy} \end{matrix}\right\} = \iint -\frac{1}{R}\left[\frac{\partial^2 S}{\partial\psi^2} - \frac{\partial S}{\partial\psi}\cot\psi\right]\begin{Bmatrix}\cos 2\alpha \\ \sin 2\alpha\end{Bmatrix}[\Delta g - \Delta g_P]\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^2 = -\frac{1}{R}\frac{\sqrt{(n+2)(n+1)n(n-1)}}{n-1} \qquad \text{for } n \geq 2$$

KERNEL:

$$-\frac{1}{R}\left(\frac{\partial^2 S}{\partial\psi^2} - \frac{\partial S}{\partial\psi}\cot\psi\right)\begin{Bmatrix}\cos 2\alpha \\ \\ \sin 2\alpha\end{Bmatrix} = -\frac{1}{R}\sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n^2(\cos\psi)\begin{Bmatrix}\cos 2\alpha \\ \\ \sin 2\alpha\end{Bmatrix}$$

REFERENCES:

Malkin, Gerlands Beiträge der Geophysik, 38 (1933), 53-63; in particular page 56, equations 8 and 8a.

REMARKS:

An analytic expression for $\frac{\partial^2 S(\psi)}{\partial\psi^2}$ is given in Reed, OSU Report #201, page 72, equation 5.37.

SECTION 5

SPHERICAL GEODETIC TRANSFORMATIONS

WITH

GRAVITY VARIATIONS AS INPUT

TRANSFORMATION: STOKES ANALOG FOR GRAVITY VARIATION

Input: dg = gravity variation

Output: $N = T/G$ = geoid height

EXPLICIT FORM:

$$N = \frac{R}{G} \iint S_{dg}(\psi)\ dg\ \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{R}{G} \begin{cases} 0 & \text{for } n = 0 \\ \frac{1}{n} & \text{for } n \geq 1 \end{cases}$$

KERNEL:

$$S_{dg}(\psi) = \sum_{n=1}^{\infty} \frac{2n+1}{n} P_n(\cos\psi) = \frac{1}{\sin\frac{\psi}{2}} - 2 - \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)$$

REFERENCES:

Pick-Picha-Vyskocil, page 476, equation 1555.

REMARKS:

The kernel is closely related to Callandreau's $G(\psi)$ function, given in Pick-Picha-Vyskocil, page 245, near equation 714.

TRANSFORMATION:

Input: dg = gravity variation

Output: Δg = gravity anomaly

EXPLICIT FORM:

$$\Delta g = dg - \frac{G}{R} N = \iint \left[\delta_{2D}(\psi) - S_{dg}(\psi) \right] dg \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{n-1}{n} = 1 - \frac{1}{n}$$

KERNEL:

$$\left[\delta_{2D}(\psi) - S_{dg}(\psi) \right] = \sum_{n=0}^{\infty} \frac{n-1}{n} (2n+1) P_n (\cos \psi)$$

formally

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: dg = gravity variation

Output: μ = surface layer density

EXPLICIT FORM:

$$\mu = dg + \frac{G}{R}\frac{N}{2} = \iint \left[\delta_{2D}(\psi) + \frac{1}{2} S_{dg}(\psi)\right] dg \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0 \\ \frac{2n+1}{2n} = 1 + \frac{1}{2n} & \text{for } n \geq 1 \end{cases}$$

KERNEL:

$$\left[\delta_{2D}(\psi) + \frac{1}{2} S_{dg}(\psi)\right] = \sum_{n=1}^{\infty} \frac{(2n+1)^2}{2n} P_n(\cos\psi) \qquad \text{formally}$$

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: dg = gravity variation

Output: δg = gravity disturbance

EXPLICIT FORM:

$$\delta g = dg + \frac{G}{R} N = \iint \left[\delta_{2D}(\psi) + S_{dg}(\psi) \right] dg \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0 \\ \frac{n+1}{n} = 1 + \frac{1}{n} & \text{for } n \geq 1 \end{cases}$$

KERNEL:

$$\left[\delta_{2D}(\psi) + S_{dg}(\psi) \right] = \sum_{n=1}^{\infty} \frac{n+1}{n} (2n+1) P_n (\cos \psi) \qquad \text{formally}$$

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: dg = gravity variation

Output: d_2g = generalized gravity anomaly of order two

EXPLICIT FORM:

$$d_2g = dg + 2\frac{G}{R}N = \iint \left[\delta_{2D}(\psi) + 2\, S_{dg}(\psi)\right] dg \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0 \\ \dfrac{n+2}{n} = 1 + \dfrac{2}{n} & \text{for } n \geq 1 \end{cases}$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION: TOTAL VENING-MEINESZ ANALOG FOR GRAVITY VARIATION

Input: dg = gravity variation

Output: $G\varepsilon$ = magnitude of horizontal gravity disturbance

EXPLICIT FORM:

$$G\varepsilon = \iint \frac{\partial S_{dg}(\psi)}{\partial \psi} \, dg \, \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{n} = \sqrt{\frac{n+1}{n}} \qquad (n \geq 1)$$

KERNEL:

$$\frac{\partial S_{dg}(\psi)}{\partial \psi} = \sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{n} \frac{(2n+1)}{\sqrt{n(n+1)}} P_n^1(\cos\psi) = \sum_{n=1}^{\infty} \frac{2n+1}{n} P_n^1(\cos\psi)$$

REFERENCES:

REMARKS:

TRANSFORMATION: VENING-MEINESZ ANALOG FOR GRAVITY VARIATION

Input: dg = gravity variation

Output: $G\vec{\epsilon}$ = horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\vec{\epsilon} = \iint \frac{\partial S_{dg}(\psi)}{\partial \psi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} dg \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{n} = \sqrt{\frac{n+1}{n}} \qquad (n \geq 1)$$

KERNEL:

$$\frac{\partial S_{dg}(\psi)}{\partial \psi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} = \sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{n} \frac{2n+1}{\sqrt{n(n+1)}} P_n^1(\cos\psi) = \sum_{n=1}^{\infty} \frac{2n+1}{n} P_n^1(\cos\psi)$$

REFERENCES:

REMARKS:

SECTION 6

SPHERICAL GEODETIC TRANSFORMATIONS

WITH

SURFACE LAYER DENSITIES AS INPUT

TRANSFORMATION: STOKES ANALOG FOR SURFACE LAYER DENSITY

Input: μ = surface layer density

Output: $N = T/G$ = geoid height

EXPLICIT FORM:

$$N = \frac{R}{G} \iint \frac{2\mu}{2 \sin \frac{\psi}{2}} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{R}{G} \frac{1}{n + \frac{1}{2}} = \frac{R}{G} \frac{2}{2n+1}$$

KERNEL:

$$\frac{2}{2 \sin \frac{\psi}{2}} = 2 \sum_{n=0}^{\infty} P_n(\cos \psi)$$

REFERENCES:

Heiskanen-Moritz, page 237, equation 6-58.
Pick-Picha-Vyskocil, page 239, equation 677.
Meissl, OSU Report #151, equation 3-9.

REMARKS:

The kernel may be called the "reciprocal distance" kernel because the function $2 \sin \frac{\psi}{2}$ is the direct linear distance between two points on a unit circle separated by a spherical distance (i.e. central angle) ψ.

TRANSFORMATION:

Input: μ = surface layer density

Output: ?

EXPLICIT FORM:

$$= \int_0^{\psi_0} \int_0^{2\pi} \frac{\sqrt{2}}{\sqrt{-\cos\psi_0 - \cos\psi}} \mu(\psi,\alpha) \frac{\sin\psi \, d\alpha \, d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{1 - \cos(n + \frac{1}{2})\psi_0}{n + \frac{1}{2}} = \left(\frac{1}{n + \frac{1}{2}}\right) 2\sin^2\left(\frac{2n+1}{4}\right)\psi_0$$

KERNEL:

$$K(\cos\psi) = \begin{Bmatrix} \dfrac{\sqrt{2}}{\sqrt{-\cos\psi_0 - \cos\psi}} & \text{for } \psi < \psi_0 \\ 0 & \text{for } \psi > \psi_0 \end{Bmatrix} = \sum_{n=0}^{\infty} \left(2\sin\frac{2n+1}{4}\psi_0\right)^2 P_n(\cos\psi)$$

REFERENCES:

See the complementary transformation on the next page.

REMARKS:

As ψ_0 approaches π, $K(\cos\psi)$ approaches $\frac{1}{\sin\frac{\psi}{2}}$. However, the transformation is not the truncated Stokes analog for the surface layer density. (The kernel itself depends on ψ_0).

TRANSFORMATION:

Input: μ = surface layer density

Output: ?

EXPLICIT FORM:

$$= \int_{\psi_0}^{\pi} \int_{0}^{2\pi} \frac{\sqrt{2}}{\sqrt{+\cos \psi_0 - \cos \psi}} \mu(\psi,\alpha) \frac{\sin\psi \, d\alpha \, d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{\cos\left(n + \frac{1}{2}\right)\psi_0}{n + \frac{1}{2}}$$

KERNEL:

$$K(\cos \psi) = \begin{Bmatrix} 0 & \text{for } \psi < \psi_0 \\ \dfrac{\sqrt{2}}{\sqrt{+\cos \psi_0 - \cos \psi}} & \text{for } \psi > \psi_0 \end{Bmatrix} = \sum_{n=0}^{\infty} 2\cos(n + \tfrac{1}{2})\psi_0 \, P_n(\cos \psi)$$

REFERENCES:

Gradstheyn-Ryzhik, page 1029, equation 8.927.

Abromawitz-Stegun, page 786, equation 22.13.10.

Morse-Feshbach, Volume II, page 1327.

Robin, Volume II, page 312, equation 111.

REMARKS:

As ψ_0 approaches 0, $K(\cos \psi)$ approaches $\frac{1}{\sin \frac{\psi}{2}}$. However, the transformation is not the residual Stokes analog for the surface layer density. (The kernel depends on ψ_0).

TRANSFORMATION:

Input: μ = surface layer density

Output: ?

EXPLICIT FORM:

$$= \iint \frac{\sqrt{2}}{\sqrt{a - \cos\psi}} \, \mu(\psi,\alpha) \, \frac{d\sigma}{4\pi} \qquad (a \geq 1)$$

SPECTRUM:

$$\lambda_n = \frac{1}{n + \frac{1}{2}} \exp\left[- \left(n + \frac{1}{2} \right) \text{Arccosh } a \right]$$

KERNEL:

$$\sqrt{\frac{2}{a - \cos\psi}} = \sum_{n=0}^{\infty} 2 \exp\left[-(n + \tfrac{1}{2}) \text{Arccosh } a \right] P_n(\cos\psi)$$

REFERENCES:

Gradstheyn-Ryzhik, page 822, equation 7.225.4.

REMARKS:

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$ = arbitrary input function

Output: $f_{OUT}(0,0)$ = transformed (output) function

EXPLICIT FORM:

$$f_{OUT}(0,0) = \iint K(\psi,\ \psi_0) f_{IN}(\psi,\alpha) \frac{d\sigma}{4\pi}$$

where $K(\psi,\ \psi_0)$ is defined under "kernel" below.

SPECTRUM:

$$\lambda_n = e^{in\,\psi_0} = \cos n\psi_0 + i \sin n\,\psi_0$$

KERNEL:

$$K(\psi,\ \psi_0) \triangleq \left\{ \begin{array}{ll} \dfrac{-\exp(-i\ \psi_0/2)}{\sqrt{2(\cos\psi - \cos\ \psi_0)}} & \text{for } 0 < \psi < \psi_0 \leq \pi \\ \\ \dfrac{+\exp(-i\ \psi_0/2)}{\sqrt{2(\cos\ \psi_0 - \cos\ \psi)}} & \text{for } 0 \leq \psi_0 < \psi < \pi \end{array} \right\} = \sum_{n=0}^{\infty} e^{in\psi_0}(2n+1)P_n(\cos\ \psi)$$

REFERENCES:

Robin, Vol. II, page 313, equation 113.

REMARKS:

TRANSFORMATION:

Input: μ = surface layer density

Output: $N_{ANTIPODE}$ = geoid height at the antipode

EXPLICIT FORM:

$$N_{ANTIPODE} = \frac{R}{G} \iint \frac{\sqrt{2}}{\sqrt{1 + \cos\ \psi}} \ \mu(\psi, \alpha) \ \frac{d\sigma}{4\pi} = \frac{R}{G} \iint \frac{2\ \mu\ (\psi, \alpha)}{2 \cos \frac{\psi}{2}} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{R}{G} \frac{2(-1)^n}{2n+1} = \frac{R}{G} \frac{(-1)^n}{n + \frac{1}{2}}$$

KERNEL:

$$\sqrt{\frac{2}{1 + \cos\ \psi}} = \frac{1}{\cos \frac{\psi}{2}} = \sum_{n=0}^{\infty} (-1)^n P_n (\cos\ \psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 476, equation 1546.

REMARKS:

TRANSFORMATION:

Input: μ = surface layer density

Output: ?

EXPLICIT FORM:

$$= \int_0^{\psi_0} \int_0^{2\pi} \frac{\sqrt{2}}{\sqrt{\cos\psi - \cos\psi_0}} \mu(\psi,\alpha) \frac{\sin\psi \, d\alpha \, d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{\sin(n+\frac{1}{2})\psi_0}{n+\frac{1}{2}}$$

KERNEL:

$$K(\cos\psi) = \begin{cases} \sqrt{\dfrac{2}{\cos\psi - \cos\psi_0}} & \text{for } \psi < \psi_0 \\ 0 & \text{for } \psi \geq \psi_0 \end{cases} = \sum_{n=0}^{\infty} 2\sin(n+\tfrac{1}{2})\psi_0 \, P_n(\cos\psi)$$

REFERENCES:

Abramowitz-Stegun, page 786, equation 22.13.11.
Morse-Feshbach, Volume II, page 1327.
Robin, Volume II, page 312, equation 112.

REMARKS:

When $\psi_0 = \pi$, then $K(\cos\psi) = \sqrt{\dfrac{2}{1+\cos\psi}} = \dfrac{1}{\cos\frac{\psi}{2}}$

TRANSFORMATION:

Input: μ = surface layer density

Output: ?

EXPLICIT FORM:

$$= \int_{\psi_0}^{\pi} \int_{0}^{2\pi} \frac{\sqrt{2}}{\sqrt{\cos\psi + \cos\psi_0}} \mu(\psi,\alpha) \frac{\sin\psi \, d\alpha \, d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{(-1)^n - \sin(n+\frac{1}{2})\psi_0}{n+\frac{1}{2}}$$

KERNEL:

$$K(\cos\psi) = \begin{cases} 0 & \text{for } \psi < \psi_0 \\ \sqrt{\dfrac{2}{\cos\psi + \cos\psi_0}} & \text{for } \psi > \psi_0 \end{cases}$$

REFERENCES:

See the complementary transformation on the previous page.

REMARKS:

When $\psi_0 = 0$, then $K(\cos\psi) = \sqrt{\dfrac{2}{1+\cos\psi}} = \dfrac{1}{\cos\frac{\psi}{2}}$

TRANSFORMATION: IDELSON

Input: μ = surface layer density

Output Δg = gravity anomaly

EXPLICIT FORM:

$$\Delta g = \mu - \frac{3}{2}\frac{G}{R} N = \iint \left[\delta_{2D}(\psi) - \frac{3}{(2\sin\frac{\psi}{2})}\right] \mu(\psi,\alpha)\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0,1 \\ \dfrac{n-1}{n+\frac{1}{2}} = \dfrac{2n-2}{2n+1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$\left[\delta_{2D}(\psi) - \frac{3}{2\sin\frac{\psi}{2}}\right] = \sum_{n=2}^{\infty} 2(n-1)P_n(\cos\psi) \qquad \text{formally}$$

REFERENCES:

Pick-Picha-Vyskocil, page 240, equation 681.

REMARKS:

TRANSFORMATION:

Input: μ = surface layer density

Output: dg = gravity variation

EXPLICIT FORM:

$$dg = \mu - \frac{1}{2}\frac{G}{R}N = \iint \left[\delta_{2D}(\psi) - \frac{1}{2\sin\frac{\psi}{2}}\right]\mu(\psi,\alpha)\,\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{n}{n+\frac{1}{2}} = \frac{2n}{2n+1}$$

KERNEL:

$$\left[\delta_{2D}(\psi) - \frac{1}{2\sin\frac{\psi}{2}}\right] = \sum_{n=0}^{\infty} 2n\,P_n\,(\cos\psi) \qquad \text{formally}$$

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: μ = surface layer density

Output: δg = gravity disturbance

EXPLICIT FORM:

$$\delta g = \mu + \frac{1}{2}\frac{G}{R} N = \iint \left[\delta_{2D}(\psi) + \frac{1}{2\sin\frac{\psi}{2}}\right] \mu(\psi,\alpha)\, \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{n+1}{n+\frac{1}{2}} = \frac{2n+2}{2n-2}$$

KERNEL:

$$\left[\delta_{2D}(\psi) + \frac{1}{2\sin\frac{\psi}{2}}\right] = \sum_{n=0}^{\infty} 2(n+1)\, P_n(\cos\psi) \qquad \text{formally}$$

REFERENCES:

Heiskanen-Moritz, page 238, equation 6-60 (related result).

REMARKS:

TRANSFORMATION: TOTAL VENING-MEINESZ ANALOG FOR SURFACE LAYER DENSITY

Input: μ = surface layer density

Output: $G\varepsilon$ = magnitude of norizontal gravity disturbance vector

EXPLICIT FORM:

$$G\varepsilon = \iint \frac{-2\cos\frac{\psi}{2}}{(2\sin\frac{\psi}{2})}\ \mu(\psi,\alpha)\ \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{n+\frac{1}{2}} \qquad (n \geq 1)$$

KERNEL:

$$\frac{-2\cos\frac{\psi}{2}}{(2\sin\frac{\psi}{2})^2} = \sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{(n+\frac{1}{2})}\ \frac{(2n+1)}{\sqrt{n(n+1)}}\ P_n^1(\cos\psi) = \sum_{n=1}^{\infty} 2\,P_n^1(\cos\psi)$$

REFERENCES:

Meissl, OSU Report #151, page 46, Table I.

REMARKS:

The explicit form of the kernel is obtained by differentiating $1/\sin\psi/2$ with respect to ψ, just as the Vening-Meinesz kernel is obtained from the Stokes kernel.

TRANSFORMATION: VENING-MEINESZ ANALOG FOR SURFACE LAYER DENSITY

Input: μ = surface layer density

Output: $G\vec{\epsilon}$ = horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\vec{\epsilon} = \begin{Bmatrix} G\xi \\ G\eta \end{Bmatrix} = \iint \frac{-2\cos\frac{\psi}{2}}{(2\sin\frac{\psi}{2})^2} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} \mu(\psi,\alpha)\,\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{n+\frac{1}{2}} \qquad (n \geq 1)$$

KERNEL:

$$\frac{-2\cos\frac{\psi}{2}}{(2\sin\frac{\psi}{2})^2} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} = \sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{(n+\frac{1}{2})} \frac{(2n+1)}{\sqrt{n(n+1)}} P_n^1(\cos\psi) \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} = \sum_{n=1}^{\infty} 2P_n^1(\cos\psi) \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix}$$

REFERENCES:

Heiskanen-Moritz, page 238, equation 6-60.

REMARKS:

TRANSFORMATION:

Input: μ = surface layer density

Output: ?

EXPLICIT FORM:

$$= \int_0^{\psi_0} \int_0^{2\pi} \frac{2^{-\frac{1}{2}} \sin \psi}{(-\cos \psi_0 - \cos \psi)^{3/2}} \; \mu(\psi,\alpha) \; \frac{\sin\psi \; d\alpha \; d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{(n + \frac{1}{2})} \left[1 - \cos(n + \tfrac{1}{2})\psi_0\right] = \frac{\sqrt{n(n+1)}}{2n+1} \left[2 \sin \left(\frac{2n+1}{4}\right)\psi_0\right]^2$$

KERNEL:

$$\left\{\begin{array}{ll} \dfrac{2^{-\frac{1}{2}} \sin \psi}{(-\cos \psi_0 - \cos \psi)^{3/2}} & \text{for } \psi < \psi_0 \\ & \\ 0 & \text{for } \psi > \psi_0 \end{array}\right\} = \sum_{n=1}^{\infty} \left[2 \sin \left(\frac{2n+1}{4}\right) \psi_0\right]^2 P_n^1(\cos \psi)$$

REFERENCES:

REMARKS:

As ψ_0 approaches π, $K(\cos \psi)$ approaches $\dfrac{2 \cos \frac{\psi}{2}}{\left(2 \sin \frac{\psi}{2}\right)^2}$

However, the transformation is not the truncated Vening-Meinesz analog for the surface layer density. (The kernel itself depends on ψ_0.)

TRANSFORMATION:

Input: μ = surface layer density

Output: ?

EXPLICIT FORM:

$$= \int_{\psi_0}^{\pi} \int_{0}^{2\pi} \frac{2^{-\frac{1}{2}} \sin \psi}{(\cos \psi_0 - \cos \psi)^{3/2}} \mu(\psi,\alpha) \frac{\sin\psi \, d\alpha \, d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{n + \frac{1}{2}} \cos \left(n + \tfrac{1}{2}\right) \psi_0 \qquad (n \geq 1)$$

KERNEL:

$$\left\{ \begin{array}{ll} 0 & \text{for } \psi < \psi_0 \\ \dfrac{2^{-\frac{1}{2}} \sin \psi}{(\cos \psi_0 - \cos \psi)^{3/2}} & \text{for } \psi > \psi_0 \end{array} \right\} = \sum_{n=1} 2 \cos \left(n + \tfrac{1}{2}\right) \psi_0 \, P_n^1(\cos \psi)$$

REFERENCES:

Erdelyi, Higher Transcendental Functions, Vol. I, page 166, equation 3.10.2.

REMARKS:

As ψ_0 approaches 0, $K(\cos \psi_0)$ approaches $\dfrac{2 \cos \frac{\psi}{2}}{\left[2 \sin \frac{\psi}{2}\right]^2}$

However, the transformation is not the residual Vening-Meinesz analog for the surface layer density. (The kernel depends on ψ_0.)

SECTION 7

SPHERICAL GEODETIC TRANSFORMATIONS WITH GRAVITY DISTURBANCES AS INPUT

TRANSFORMATION: NEUMANN, STOKES ANALOG FOR GRAVITY DISTURBANCE

Input: δg = gravity disturbance

Output: N = geoid height

EXPLICIT FORM:

$$N = \frac{R}{G} \iint S_{\delta g}(\psi)\, \delta g \, \frac{d\sigma}{4\pi}$$

where $S_{\delta g}(\psi)$ is defined in "Kernel" section below.

SPECTRUM:

$$\lambda_n = \frac{R}{G} \frac{1}{(n+1)}$$

KERNEL: (Neumann Function)

$$S_{\delta g}(\psi) = \frac{1}{\sin \frac{\psi}{2}} - \ell n \left(\frac{1 + \sin \frac{\psi}{2}}{\sin \frac{\psi}{2}} \right) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos \psi)$$

REFERENCES:

Heiskanen-Moritz, page 36, equation 1-91.

Pick-Picha-Vyskocil, pages 489-495; page 477, equation 1557.

REMARKS:

TRANSFORMATION:

Input: δg = gravity disturbance

Output: Δg = gravity anomaly

EXPLICTI FORM:

$$\Delta g = \delta g - 2\frac{G}{R} N = \iint \left[\delta_{2D}(\psi) - 2S_{\delta g}(\psi)\right] \delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0,1 \\ \frac{n-1}{n+1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$\left[\delta_{2D}(\psi) - 2S_{\delta g}(\psi)\right] = \sum_{n=2}^{\infty} \frac{n-1}{n+1} (2n+1) P_n (\cos \psi)$$

formally

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: δg = gravity disturbance

Output: dg = gravity variation

EXPLICIT FORM:

$$dg = \delta g - \frac{G}{R} N = \iint \left[\delta_{2D}(\psi) - S_{\delta g}(\psi)\right] \delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{n}{n+1}$$

KERNEL:

$$\left[\delta_{2D}(\psi) - S_{\delta g}(\psi)\right] = \sum_{n=0}^{\infty} \frac{n}{n+1} (2n+1)\, P_n (\cos \psi)$$

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: δg = gravity disturbance

Output: μ = surface layer density

EXPLICIT FORM:

$$\mu = \delta g - \frac{1}{2}\frac{G}{R} N = \iint \left[\delta_{2D}(\psi) - \frac{1}{2} S_{\delta g}(\psi)\right] \delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{n + \frac{1}{2}}{n+1} = \frac{2n+1}{2(n+1)}$$

KERNEL:

$$\left[\delta_{2D}(\psi) - \frac{1}{2} S_{\delta g}(\psi)\right] = \sum_{n=0}^{\infty} \frac{(2n+1)^2}{2(n+1)} P_n(\cos\psi)$$ formally

REFERENCES:

REMARKS:

TRANSFORMATION: TOTAL HILBERT

Input: δg = gravity disturbance

Output: $G\epsilon$ = magnitude of horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\epsilon = \iint \frac{\partial S_{\delta g}(\psi)}{\partial \psi} \delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{n+1} = \sqrt{\frac{n}{n+1}}$$

KERNEL:

$$\frac{\partial S_{\delta g}(\psi)}{\partial \psi} = \frac{-\cos\frac{\psi}{2}}{\left[2\sin^2\frac{\psi}{2}\right]\left[1+\sin\frac{\psi}{2}\right]} = \sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{n+1} \frac{(2n+1)}{\sqrt{n(n+1)}} P_n^1(\cos\psi) =$$

$$= \sum_{n=1}^{\infty} \frac{(2n+1)}{(n+1)} P_n^1(\cos\psi)$$

REFERENCES:

REMARKS:

TRANSFORMATION: HILBERT, VENING-MEINESZ ANALOG

Input: δg = gravity disturbance

Output: $G\vec{\varepsilon}$ = horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\vec{\varepsilon} = \iint \frac{\partial S_{\delta g}(\psi)}{\partial \psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} \delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \frac{\sqrt{n(n+1)}}{n+1} = \sqrt{\frac{n}{n+1}}$$

KERNEL:

$$\frac{\partial S_{\delta g}(\psi)}{\partial \psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} \quad \sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{n+1} \frac{2n+1}{\sqrt{n(n+1)}} P_n^1 (\cos \psi) \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} =$$

$$= \sum_{n=1}^{\infty} \frac{2n+1}{n+1} P_n^1 (\cos \psi) \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix}$$

REFERENCES:

REMARKS:

The name of Hilbert is associated with this transformation because the transformation is the analog of the planar two-dimensional Hilbert transform.

TRANSFORMATION:

Input: δg = gravity disturbance

Output: $\frac{\partial\,\delta g}{\partial r} = -T_{zz}$ = vertical stress gravity gradient

EXPLICIT FORM:

$$\frac{\partial\,\delta g}{\partial r} = -\frac{2}{R}\,\delta g_P + \frac{2}{R}\iint \frac{(\delta g - \delta g_P)}{\left[2\sin\frac{\psi}{2}\right]^3}\,\frac{d\sigma}{4\pi}$$

$$= \iint \frac{\partial^2 S_{\delta g}(r,\psi)}{\partial r^2}\,\delta g\,\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = -\frac{1}{R}(n+2)$$

KERNEL:

REFERENCES:

REMARKS:

This is the same transformation as:

- the one transforming N into d_2g,
- the one transforming Δg into $\frac{\partial\,\Delta g}{\partial r}$.

TRANSFORMATION:

Input: δg = gravity disturbance

Output: $\left\{\begin{array}{ll} T_{zx} = \partial\,\delta g/\partial\psi & \text{where } \alpha = 0 \\ T_{zy} = \partial\,\delta g/\partial\psi & \text{where } \alpha = 90° \end{array}\right\}$ = vertical shear gravity gradients

EXPLICIT FORM:

$$\left.\begin{array}{l} T_{zx} \\ \\ T_{zy} \end{array}\right\} = \iint \frac{\partial^2 S_{\delta g}(r,\psi)}{\partial r\ \partial\psi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} \delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = -\frac{(n+2)}{R}\frac{\sqrt{n(n+1)}}{n+1}$$

KERNEL:

REFERENCES:

REMARKS:

By convention, $x \leftrightarrow$ local north
$y \leftrightarrow$ local east
$z \leftrightarrow$ local down
$r \leftrightarrow$ local up

TRANSFORMATION:

Input: δg = gravity disturbance

Output: $\begin{cases} -(T_{yy} - T_{xx}) = \text{difference of horizontal stress gradients} \\ T_{xy} = \text{horizontal shear gradient} \end{cases}$

EXPLICIT FORM:

$$\left.\begin{matrix} -(T_{yy} - T_{xx}) \\ 2T_{xy} \end{matrix}\right\} = -\frac{1}{R} \iint \left[\frac{\partial^2 S_{\delta g}(\psi)}{\partial \psi^2} - \cot \psi \frac{\partial S_{\delta g}(\psi)}{\partial \psi} \right] \left\{ \begin{matrix} \cos 2\alpha \\ \sin 2\alpha \end{matrix} \right\} \delta g \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^2 = \frac{-1}{R} \frac{\sqrt{(n+2)(n+1)\,n\,(n-1)}}{n+1} \qquad \text{for } n \geq 2$$

KERNEL:

$$\left(\frac{\partial^2 S_{\delta g}(\psi)}{\partial \psi^2} - \cot\psi \frac{\partial S_{\delta g}}{\partial \psi} \right) \left\{ \begin{matrix} \cos 2\alpha \\ \sin 2\alpha \end{matrix} \right\} = \sum_{n=2}^{\infty} \frac{(2n+1)}{n+1} P_n^2(\cos \psi) \left\{ \begin{matrix} \cos 2\alpha \\ \sin 2\alpha \end{matrix} \right\}$$

REFERENCES:

Malkin, Gerlands Beiträge der Geophysik, 38 (1933), 53-60, in particular page 56, equations 8 and 8a modified for δg in place of Δg.

REMARKS:

SECTION 8

SPHERICAL GEODETIC TRANSFORMATIONS

WITH

THE OUTWARD PARTIAL OF GEOID HEIGHT

OR

OUTWARD DEFLECTION AS INPUT

TRANSFORMATION: MALKIN

Input: $\partial N/\partial\psi = R\vec{\varepsilon}\cdot(\cos\alpha, \sin\alpha)$ = outward partial of geoid height

Output: $N = T/G$ = geoid height

EXPLICIT FORM:

$$N = \iint \left(-\cot\frac{\psi}{2}\right) \frac{\partial N}{\partial\psi} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = -W_n = \begin{cases} -\dfrac{\pi}{2}\left[\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots\ (n)}\right]^2 & (n \text{ even}) \\ \\ -\dfrac{\pi}{2}\left[\dfrac{1\cdot 3\cdot 5\cdots\ (n)}{2\cdot 4\cdot 6\cdots(n+1)}\right]^2 \left(\dfrac{n+1}{n}\right) & (n \text{ odd}) \end{cases}$$

KERNEL:

$$-\cot\frac{\psi}{2} = \sum_{n=0}^{\infty} (-W_n)(2n+1)\, P_n(\cos\psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 245, equation 713.

REMARKS:

- A fortiori, this transformation must be the same as Callandreau's transformation, although their equivalence has not yet been proven.
- The W_n are called the Wallis coefficients.

TRANSFORMATION: CALLANDREAU

Input: $\partial N/\partial\psi$ = outward partial of geoid height

Output: $N = T/G$ = geoid height

EXPLICIT FORM:

$$N = -\frac{\partial N}{\partial \psi}\bigg|_{\text{ORIGIN}} + \iint \frac{\partial G(\psi)}{\partial \psi}\,\frac{\partial N}{\partial \psi}\,\frac{d\sigma}{4\pi}$$

$$\text{where } G(\psi) = \frac{1}{\sin\frac{\psi}{2}} - 2 - 3\cos\psi - \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)$$

SPECTRUM:

$\lambda_n = -W_n$ (a fortiori)

(See Malkin's transformation)

KERNEL:

$$\left[-\delta_{2D}(\psi) + \frac{\partial G(\psi)}{\partial \psi}\right] \approx -\cot\frac{\psi}{2} \qquad \text{formally}$$

REFERENCES:

Pick-Picha-Vyskocil, page 245, equation 714.

REMARKS:

$$\frac{\partial G(\psi)}{\partial \psi} = \frac{\cos\frac{\psi}{2}}{(2\sin^2\frac{\psi}{2})(1+\sin\frac{\psi}{2})}\left[-1-2\sin\frac{\psi}{2} - 2\sin^2\frac{\psi}{2} + 12\sin^3\frac{\psi}{2} + 12\sin^4\frac{\psi}{2}\right]$$

TRANSFORMATION:

Input: $\partial N/\partial\psi$ = outward partial of geoid height

Output: $N_{ANTIPODE}$ = geoid height at the antipode

EXPLICIT FORM:

$$N_{ANTIPODE} = \iint \left(\tan \frac{\psi}{2}\right) \frac{\partial N}{\partial \psi} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} +W_n & \text{for n even} \\ -W_n & \text{for n odd} \end{cases}$$

KERNEL:

$$\tan \frac{\psi}{2} = \sum_{n=0}^{\infty} (-1)^n W_n (2n+1) P_n (\cos \psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 245, equation 712.

REMARKS:

TRANSFORMATION:

Input: $\partial N/\partial\psi$ = outward partial of *geoid height*

Output: $-\frac{1}{2}\left[N - N_{ANTIPODE}\right]$

EXPLICIT FORM:

$$-\frac{1}{2}\left[N - N_{ANTIPODE}\right] = \iint \frac{1}{\sin\psi} \frac{\partial N}{\partial\psi} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} W_n & \text{for n even} \\ 0 & \text{for n odd} \end{cases}$$

KERNEL:

$$\frac{1}{\sin\psi} = \sum_{k=0}^{\infty} [2(2k)+1]\ W_{2k}\ P_{2k}\ (\cos\ \psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 245, equation 710.

Gradshteyn-Ryzhik, page 1027, equation 8.921.3,
page 1029, equation 8.925.4.

Robin, Vol. II, page 310, equation 98.

Morse-Feshbach, Vol. II, page 1326.

REMARKS:

TRANSFORMATION: MOLODENSKII #2, MALKIN

Input: $\partial \Delta g/\partial\psi$ = outward partial of gravity anomaly

Output: Δg = gravity anomaly

EXPLICIT FORM:

$$\Delta g = \iint \left(-\cot \frac{\psi}{2}\right) \frac{\partial \Delta g}{\partial \psi} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = -W_n$$

KERNEL:

$$-\cot \frac{\psi}{2} = \sum_{n=0}^{\infty} (-W_n)(2n+1)\, P_n(\cos\psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 245, equation 714½.

REMARKS:

This is the same transformation as the one transforming $\frac{\partial N}{\partial \psi}$ into N.

TRANSFORMATION: MIGAL

Input: $\partial \Delta g/\partial \psi$ = outward partial of gravity anomaly

Output: N = geoid height

EXPLICIT FORM:

$$N = -\frac{R}{G}\iint \frac{2\Phi(\psi)}{\sin\psi}\,\frac{\partial\,\Delta g}{\partial\psi}\,\frac{d\sigma}{4\pi}$$

$$\text{where } \Phi(\psi) = \frac{1}{2}\int_0^{\psi} S(\psi')\sin\psi'\,d\psi'$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n = 0,1 \\ -\frac{R}{G}\,\frac{W_n}{n-1} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

REFERENCES:

Pick-Picha-Vyskocil, page 279, equation 892.

Migal, C.R. Acad. Sci. USSR, 16 (1937), page 169 ff.

REMARKS:

An explicit expression for $\Phi(\psi)$ is given in Pick-Picha-Vyskocil, page 478, equation 1574.

TRANSFORMATION: MAGNITSKII

Input: $\partial N/\partial\psi$ = outward partial of geoid height

Output: Δg = gravity anomaly

EXPLICIT FORM:

$$\Delta g = \frac{G}{R}\iint\left[-\tan\frac{\psi}{2} + \frac{1}{\sin\psi}\left(\frac{3}{2} - \frac{1}{2\sin\frac{\psi}{2}}\right)\right]\frac{\partial N}{\partial\psi}\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = -\frac{G}{R}(n-1)\,W_n$$

KERNEL:

REFERENCES:

Pick-Picha-Vyskocil, page 246, equation 719.
Molodenskii, page 52, equation III.2.11.

REMARKS:

TRANSFORMATION:

Input: $\partial N/\partial\psi$ = outward partial of geoid height

Output: dg = gravity variation

EXPLICIT FORM:

SPECTRUM:

$$\lambda_n = -\frac{G}{R} n W_n$$

KERNEL:

REFERENCES:

REMARKS:

$W_n = O(1/n)$. Hence $\lim_{n\to\infty} \lambda_n = 1$

TRANSFORMATION:

Input: $\partial N/\partial \psi$ = outward partial of geoid height

Output: μ = surface layer density

EXPLICIT FORM:

SPECTRUM:

$$\lambda_n = -\frac{G}{R}(n + \frac{1}{2})\tilde{W}_n$$

KERNEL:

REFERENCES:

REMARKS:

TRANSFORMATION:

Input: $\partial N/\partial \psi$ = outward partial of geoid height

Output: δg = gravity anomaly

EXPLICIT FORM:

$$\delta g = \frac{G}{R} \iint \left[\tan\frac{\psi}{2} - \frac{1}{\sin\psi} \left(1 + \frac{1}{\sin\frac{\psi}{2}} \right) \right] \frac{\partial N}{\partial \psi} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = -\frac{G}{R}(n+1)\, W_n$$

KERNEL:

REFERENCES:

REMARKS:

The explicit form is derived from the explicit forms for Δg and N in the equation $\delta g = \Delta g + \frac{2G}{R} N$.

TRANSFORMATION:

Input: $\partial N/\partial\psi$ = outward partial of geoid height

Output: $G\varepsilon$ = magnitude of horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\varepsilon = \frac{G}{R}\iint \frac{-2}{(2\sin\frac{\psi}{2})^2}\frac{\partial N}{\partial\psi}\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = -\frac{G}{R}\sqrt{n(n+1)}\;W_n \approx -\frac{G}{R}$$

KERNEL:

$$\frac{-2}{(2\sin\frac{\psi}{2})^2} = \frac{\partial}{\partial\psi}(-\cot\frac{\psi}{2}) = \sum_{n=1}^{\infty} -W_n\sqrt{n(n+1)}\frac{(2n+1)}{\sqrt{n(n+1)}}P_n^1(\cos\psi)$$

$$= \sum_{n=1}^{\infty}(-W_n)(2n+1)\,P_n^1(\cos\psi)$$

REFERENCES:

REMARKS:

It can be shown that the even and odd values of W_n bracket the quantities $1/\sqrt{n(n+1)}$ more and more closely with increasing n, and that $[1 - \sqrt{n(n+1)}\,W_n] = \frac{1}{4n^2} + O\left(\frac{1}{n^3}\right)$ for even n.

SECTION 9

SPHERICAL GEODETIC TRANSFORMATIONS

WITH

MISCELLANEOUS INPUTS

TRANSFORMATION: INVERSE VENING-MEINESZ, MALKIN

Input: ($G\xi$ or $G\eta$) = one horizontal disturbance component

Output: Δg = gravity disturbance

EXPLICIT FORM:

$$\Delta g = \iint \left[\frac{-\cos\frac{\psi}{2}}{2\sin^2\frac{\psi}{2}\left(1+\sin\frac{\psi}{2}\right)} + \cot\frac{\psi}{2} \right] (G\xi) \frac{d\sigma}{4\pi}$$

$$\Delta g = \iint \left[\frac{-\cos\frac{\psi}{2}}{2\sin^2\frac{\psi}{2}\left(1+\sin\frac{\psi}{2}\right)} + \cot\frac{\psi}{2} \right] (G\eta) \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^1 = \begin{cases} 0 & n = 0,1 \\ \dfrac{n-1}{\sqrt{n(n+1)}} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$\left[\frac{-\cos\frac{\psi}{2}}{2\sin^2\frac{\psi}{2}\left(1+\sin\frac{\psi}{2}\right)} + \cot\frac{\psi}{2} \right] = \sum_{n=2}^{\infty} \frac{(n-1)}{\sqrt{n(n+1)}} \frac{(2n+1)}{\sqrt{n(n+1)}} P_n^1(\cos\psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 246, equation 717.

Malkin, Gerlands Beiträge der Geophysik, 38 (1933), page 60, equation 15.

Meissl, OSU Report #151, page 46, Table I.

REMARKS:

The kernel is equal to $\frac{\partial K(\psi)}{\partial \psi}$ where $K(\psi) = \dfrac{1}{\sin\frac{\psi}{2}} + \ln\left[\dfrac{\sin^3\frac{\psi}{2}}{1+\sin\frac{\psi}{2}}\right]$ = Malkin's function. Note that the last two lines on page 59 of Malkin's article are in error.

TRANSFORMATION:

Input: $\partial\mu/\partial r$ = vertical gradient of surface layer density

Output: N = geoid height

EXPLICIT FORM:

$$N = \frac{R^2}{G} \iint \left[+1 - 2\sin\frac{\psi}{2} - \cos\psi \,\ell n\left(1 + \csc\frac{\psi}{2}\right)\right] \frac{\partial\mu}{\partial r} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{R^2}{G} \frac{-1}{(n + \frac{1}{2})(n + 2)}$$

KERNEL:

$$1 - 2\sin\frac{\psi}{2} - \cos\psi \,\ell n\left(1 + \csc\frac{\psi}{2}\right) = \sum_{n=0}^{\infty} \frac{-2}{(n+2)} P_n(\cos\psi)$$

REFERENCES:

Pick-Picha-Vyskocil, page 476, equation 1552.

REMARKS:

TRANSFORMATION: MORITZ

Input: $\partial\,\delta g/\partial r = -T_{zz}$ = vertical gradient of gravity disturbance

Output: N = geoid height

EXPLICIT FORM:

$$N = \frac{R^2}{G}\iint S_{MORITZ}(\psi)\frac{\partial\,\delta g}{\partial r}\frac{d\sigma}{4\pi}$$

where $S_{MORITZ}(\psi) = (4 - 12\sin^2\frac{\psi}{2})\,\ell n(1 + \csc\frac{\psi}{2}) - 6 + 12\sin\frac{\psi}{2}$

SPECTRUM:

$$\lambda_n = \frac{R^2}{G}\frac{(-1)}{(n+1)(n+2)}$$

KERNEL:

$$S_{MORITZ}(\psi) = \sum_{n=0}^{\infty}\frac{-(2n+1)}{(n+1)(n+2)}P_n(\cos\psi)$$

REFERENCES:

Moritz, OSU Report #92

Schwarz, OSU Report #242, pages 6 & 7, equations 2.5 and 2.6.

REMARKS:

Schwarz states that the effect of the remote zones in this transformation is stronger than in Stokes' Integral.

TRANSFORMATION:

Input: $\frac{\partial \delta g}{\partial \psi}$ = outward partial of gravity anomaly

$\frac{\partial \Delta g}{\partial \alpha}$ = tangential partial of gravity anomaly

Output: $G\vec{\varepsilon}$ = horizontal gravity disturbance vector

EXPLICIT FORM:

$$G\vec{\varepsilon} = \begin{Bmatrix} G\xi \\ G\eta \end{Bmatrix} = \iint S(\psi)\left[\frac{\partial \Delta g}{\partial \psi}\begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} - \frac{\partial \Delta g}{\partial \alpha}\begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix} \cot\psi\right]\frac{d\sigma}{4\pi}$$

SPECTRUM:

KERNEL:

REFERENCES:

Pick-Picha-Vyskocil, page 260, equation 793.

REMARKS:

TRANSFORMATION:

Input: T_{xx}, T_{yy} = horizontal stress gradients

Output: $\frac{\partial \delta g}{\partial r} = -T_{zz}$ = vertical gradient of gravity disturbance

EXPLICIT FORM:

$$\frac{\partial \delta g}{\partial r} = -T_{zz} = T_{xx} + T_{yy} \approx -G\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y}\right)$$

SPECTRUM:

KERNEL:

REFERENCES:

Heiskanen-Moritz, page 117, equation 2-221.

REMARKS:

Coordinate system convention:

x ↔ local north ($\alpha = 0$)
y ↔ local east ($\alpha = 90°$)
z ↔ local down
r ↔ local up

TRANSFORMATION: INVERSE OF RADIAL LAPLACIAN

Input: $\nabla^2_{RADIAL}\, T = T_{rr} = T_{zz} = -(T_{xx} + T_{yy}) = -\nabla^2_{SURFACE}\, T$

Output: T = anomalous potential

EXPLICIT FORM:

$$T = R^2 \iint 2\,\ln\left(\frac{1}{\sin\frac{\psi}{2}}\right)\left(\nabla^2_{RADIAL}\, T\right)\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = R^2 \begin{cases} 1 & \text{for } n = 0 \\ \dfrac{1}{n(n+1)} & \text{for } n \geq 1 \end{cases}$$

KERNEL:

$$2\,\ln\left(\frac{1}{\sin\frac{\psi}{2}}\right) = 1 + \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\, P_n\,(\cos\,\psi)$$

REFERENCES:

Malkin, Gerlands Beiträge der Geophysik, 38 (1938), page 63, equation 18. (Note that Malkin's symbol T represents our $\nabla^2_{RADIAL}\, T$).

Meissl, OSU Report #151, page 12, equation 2-15; page 22, equation 3-12.

Robin, Vol. II, page 311, equation 104.

REMARKS:

- Note that $\frac{\partial}{\partial\psi}\, 2\,\ln\left(\frac{1}{\sin\frac{\psi}{2}}\right) = -\cot\frac{\psi}{2}$

TRANSFORMATION: MALKIN

Input: $\nabla^2_{RADIAL} T = T_{rr} = +T_{zz}$

Output: Δg

EXPLICIT FORM:

$$\Delta g = R \iint K_{MALKIN}(\psi) \left(\nabla^2_{RADIAL} T\right) \frac{d\sigma}{4\pi}$$

$$\text{where } K_{MALKIN}(\psi) = \frac{1}{\sin\frac{\psi}{2}} + \ell n \left(\frac{\sin^3 \frac{\psi}{2}}{1 + \sin\frac{\psi}{2}}\right)$$

SPECTRUM:

$$\lambda_n = R \begin{cases} 0 & \text{for } n = 0,1 \\ \frac{n-1}{n(n+1)} & \text{for } n \geq 2 \end{cases}$$

KERNEL:

$$K_{MALKIN}(\psi) = \sum_{n=2}^{\infty} \frac{(n-1)}{n(n+1)} (2n+1) P_n(\cos\psi)$$

REFERENCES:

Malkin, Gerlands Beiträge der Geophysik, 38 (1933), page 59, equation 14.

REMARKS:

There are several errors on page 59 of Malkin's paper. After equation 14, the last of the three summations in the definition of the K functions should have a factor of two. And the last two equations on the page are erroneous.

SECTION 10

SPHERICAL MATHEMATICAL TRANSFORMATIONS

TRANSFORMATION: AVERAGING OVER A SPHERICAL CAP

Input: $f(\psi,\alpha)$ = any quantity

Output: $\bar{f}$ = average of $f(\psi,\alpha)$ over a spherical cap of spherical radius ψ_0

EXPLICIT FORM:

$$\bar{f} = \int_0^{\psi_0} \int_0^{2\pi} \frac{2}{1-\cos\psi_0} \; f(\psi,\alpha) \; \frac{\sin\psi \, d\alpha \, d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{1}{1-\cos\psi_0} \int_{\cos\psi_0}^{1} P_n(x)\,dx = \frac{P_{n-1}(\cos\psi_0) - P_{n+1}(\cos\psi_0)}{(1-\cos\psi_0)(2n+1)}$$

where $P_{-1}(\cos\psi_0) \equiv 1$ (i.e., $\lambda_0 = 1$)

KERNEL:

$$K(\cos\psi) = \begin{cases} \dfrac{2}{1-\cos\psi_0} & \text{for } \psi \le \psi_0 \\ 0 & \text{for } \psi > \psi_0 \end{cases}$$

REFERENCES:

Meissl, OSU Report #151, page 23, equation 3-14

Gradshteyn-Ryzhik, page 794, equation 7.111.

REMARKS:

- The surface area of a spherical cap of spherical radius ψ_0 is $\left(\frac{1-\cos\psi_0}{2}\right) 4\pi$. Hence, the form of the kernel.
- The spectrum $\lambda_n = O(n^{-3/2})$

TRANSFORMATION: AVERAGING OVER SPHERICAL RING

Input: $f(\psi,\alpha)$ = any quantity

Output: $\bar{f}\Big|_{\psi_0}^{\psi_1}$ = average of $f(\psi,\alpha)$ over the spherical ring having inner radius ψ_0 and outer radius ψ_1.

EXPLICIT FORM:

$$\bar{f}\Big|_{\psi_0}^{\psi_1} = \int_{\psi_0}^{\psi_1}\int_0^{2\pi} \frac{2}{\cos\psi_0 - \cos\psi_1} f(\psi,\alpha) \frac{\sin\psi \, d\alpha \, d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{1}{\cos\psi_0 - \cos\psi_1} \int_{\cos\psi_1}^{\cos\psi_0} P_n(x)\,dx =$$

$$\frac{\left[P_{n+1}(\cos\psi_0) - P_{n+1}(\cos\psi_1)\right] - \left[P_{n-1}(\cos\psi_0) - P_{n-1}(\cos\psi_1)\right]}{(\cos\psi_0 - \cos\psi_1)(2n+1)}$$

KERNEL:

$$K(\cos\psi) = \begin{cases} 0 & \text{for } \psi < \psi_0 \\ \dfrac{2}{\cos\psi_0 - \cos\psi_1} & \text{for } \psi_0 \le \psi \le \psi_1 \\ 0 & \text{for } \psi < \psi_1 \end{cases}$$

REFERENCES:

REMARKS:

- The surface area of a spherical ring having inner radius ψ_0 and outer radius ψ_1 is

$$\left(\frac{\cos\psi_0 - \cos\psi_1}{2}\right) 4\pi .$$

Hence the form of the kernel.

TRANSFORMATION:

Input: $f_{IN}(\psi, \alpha)$

Output: $f_{OUT}(0,0)$

EXPLICIT FORM:

$$f_{OUT} = \iint \ell n\left(\frac{1}{\sin\psi}\right) f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} (1 - \ell n\, 2) & \text{for } n = 0 \\ \frac{1}{n(n+1)} & \text{for } n \text{ even and non-zero} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

KERNEL:

$$\ell n\left(\frac{1}{\sin\psi}\right) = (1 - \ell n\, 2) + \sum_{k=1}^{\infty} \frac{2(2k)+1}{(2k)(2k+1)} P_{2k}(\cos\psi)$$

REFERENCES:

Robin, Vol. II, page 311, equation 106.

REMARKS:

Since $\ell n(1/\sin\psi) = -\ell n(\sin\psi)$, the spectrum associated with the kernel $\ell n(\sin\psi)$ is the negative of spectrum shown above.

TRANSFORMATION: DIRECT DISTANCE KERNEL

Input: $f_{IN}(\psi,\alpha)$ = unknown input function

Output: f_{OUT} = resulting output function

EXPLICIT FORM:

$$f_{OUT} = \iint (2 \sin\frac{\psi}{2})\, f_{IN}(\psi,\alpha) \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \frac{-2}{(2n+1)^2(n-\frac{3}{2})} = \frac{-\frac{1}{2}}{(n+\frac{1}{2})^2(n-\frac{3}{2})}$$

KERNEL:

$$(2 \sin\frac{\psi}{2}) = \sum_{n=0}^{\infty} \frac{-1}{(n+\frac{1}{2})(n-\frac{3}{2})} P_n(\cos\psi)$$

REFERENCES:

Meissl, OSU Report #151, page 22, equation 3-13.

REMARKS:

The kernel $(2 \sin\frac{\psi}{2})$ equals the direct linear distance between two points on a unit sphere separated by a spherical distance (i.e., central angle) ψ.

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint (1-\cos\psi)^{\alpha-1}(1+\cos\psi)^{\beta-1} f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi}$$

where Re $\alpha > 0$ and Re $\beta > 0$.

SPECTRUM:

$$\lambda_n = \frac{2^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ {}_3F_2\ (-n, n+1, \alpha;\ 1, \alpha+\beta;\ 1)$$

KERNEL:

REFERENCES:

Erdelyi (Transforms), Vol. II, page 276, equation 16.1.16.

REMARKS:

A collection of formulas for the explicit evaluation of the ${}_3F_2$ function with unit argument as well as further references to the literature is given in section 5.2.4 of Luke, Mathematical Functions and Their Approximations, Academic Press, 1975 (page 163 ff).

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint (\sin \tfrac{\psi}{2})^{2\alpha-2} (\cos \tfrac{\psi}{2})^{2\beta-2} f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi}$$

where Re $\alpha > 0$ and Re $\beta > 0$.

SPECTRUM:

$$\lambda_n = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \, {}_3F_2 \, (-n,n+1,\alpha;\ 1,\alpha+\beta;\ 1)$$

KERNEL:

REFERENCES:

Erdelyi (Transforms), Vol. II, page 276, equation 16.1.16.

REMARKS:

A collection of formulas for the explicit evaluation of the ${}_3F_2$ function with unit argument as well as further references to the literature is given in section 5.2.4 of Luke, Mathematical Functions and Their Approximations, Academic Press, 1975 (page 163 ff).

TRANSFORMATION: NEUMANN-STIELTJES

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint (1 - \cos\psi)^{\gamma} \; f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi}$$

where $\gamma > -3/4$

SPECTRUM:

$$\lambda_n = \frac{2^{\gamma}(\gamma!)^2(-1)^n}{(\gamma-n)!\,(\gamma+n+1)!} = \frac{2^{\gamma}\,\gamma!}{(-\gamma-1)!}\;\frac{(n-\gamma-1)!}{(n+\gamma+1)!}$$

KERNEL:

REFERENCES:

Robin, Vol. II, page 309, equation 93.

REMARKS:

Note that $(1 - \cos\psi) = 2\sin^2\frac{\psi}{2}$

TRANSFORMATION:

Input: $f_{IN}(\psi, \alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint (1 + \cos \psi)^{\gamma} \; f_{IN}(\psi, \alpha) \frac{d\sigma}{4\pi}$$

where $\gamma > -1$

SPECTRUM:

$$\lambda_n = \frac{2^{\gamma} (\gamma!)^2}{(\gamma-n)!(\gamma+n+1)!}$$

KERNEL:

REFERENCES:

Robin, Vol. II, page 309, equation 94.
Erdelyi (Transforms), Vol. II, page 316, equation 18.1.15.

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint \psi\, f(\psi,\alpha)\, \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} W_0 & \text{for } n = 0 \\ 0 & \text{for } n \text{ even and non-zero} \\ \dfrac{-W_n}{n^2} & \text{for } n \text{ odd} \end{cases}$$

KERNEL:

$$\psi = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{W_{2k+1}}{(2k+1)^2} [2(2k+1)+1]\, P_{2k+1}(\cos\psi)$$

REFERENCES:

Gradshteyn-Ryzhik, page 1029, equation 8.925.1.

REMARKS:

W_n is the Wallis coefficient.

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint \arcsin(\cos\psi)\ f_{IN}\ (\psi,\alpha)\ \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for n even} \\ \dfrac{W_n}{n(n+1)} & \text{for n odd} \end{cases}$$

KERNEL:

$$\arcsin(\cos\psi) = \sum_{k=0}^{\infty} \frac{W_{2k+1}}{(2k+1)(2k+2)} [2(2k+1)+1]\ P_{2k+1}\ (\cos\psi)$$

REFERENCES:

Gradshteyn-Ryzhik, page 825, equation 7.249.1.

Robin, Vol. II, page 311, equation 101.

REMARKS:

$$\arcsin(\cos\psi) = \sum_{k=0}^{\infty} W_{2k}\left[P_{2k+1}(\cos\psi) - P_{2k-1}(\cos\psi)\right]$$

TRANSFORMATION:

Input: $f_{IN}(\psi, \alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint \sin\psi \, f_{IN}(\psi, \alpha) \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ -\frac{W_n}{(n-1)(n+2)} & \text{for } n \text{ even and non-zero} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

KERNEL:

$$\sin\psi = \frac{\pi}{4} - \sum_{k=1}^{\infty} \frac{W_{2k}}{(2k-1)(2k+2)} [2(2k)+1] P_{2k}(\cos\psi)$$

REFERENCES:

Gradshteyn-Ryzhik, page 1028, equation 8.922.5;
page 1029, equation 8.925.2.

Robin, Vol. I, page 27-30; Vol. II, page 309, equation 97.

REMARKS:

The W_n are the Wallis coefficients.

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint (\sin\psi)^{2\gamma} f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi} \qquad \text{where } \gamma > -\frac{3}{4}$$

SPECTRUM:

$$\lambda_n = \begin{cases} \dfrac{\gamma!\,(\frac{n}{2}-\gamma-1)!\,(\frac{n-1}{2})!}{2(\frac{n}{2})!\,(-\gamma-1)!\,(\gamma+\frac{n+1}{2})!} & \text{for n even} \\ 0 & \text{for n odd} \end{cases}$$

KERNEL:

REFERENCES:

Robin, Vol. II, page 309, equation 96.

REMARKS:

- If γ is integral, the λ_n are zero for $n > \gamma$.
- If $\gamma = -\frac{1}{2}$, then $\lambda_n = W_n$ for n even and $f_{IN}(\psi,\alpha) = \frac{\partial N}{\partial \psi}$, $f_{OUT} = -\frac{1}{2}(N - N_{ANTIPODE})$.

  See this transformation for further references.

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: $\vec{f}_{OUT}$

EXPLICIT FORM:

$$\vec{f}_{OUT} = \iint (\sin\psi)^N \, f_{IN}(\psi,\alpha) \begin{Bmatrix} \cos N \\ \sin N \end{Bmatrix} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n^N = \begin{cases} 0 & \text{for } n \neq N \\ \dfrac{(-1)^N}{2N+1} \dfrac{2}{\pi} \left(W_{2N}\right)^{-\frac{1}{4}} & \text{for } n = N \end{cases}$$

KERNEL:

$$(\sin\psi)^N = \frac{(-1)^N 2^N N!}{(2N)!} P_N^N(\cos\psi) = \frac{(-1)^N 2^N N!}{(2N+1)\sqrt{(2N)!}} (2N+1) \frac{\sqrt{(N-N)!}}{\sqrt{(N+N)!}} P_N^N(\cos\psi)$$

REFERENCES:

Magnus-Oberhettinger (1949), page 64.

REMARKS:

TRANSFORMATION:

Input: $f_{IN}(\psi, \alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \int_0^{\psi_0} \int_0^{2\pi} \cos\psi \, f_{IN}(\psi, \alpha) \, \frac{\sin\psi \, d\alpha d\psi}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} \dfrac{\sin\psi_0}{4} & \text{for } n = 0 \\ \dfrac{1 - \cos^3\psi_0}{6} & \text{for } n = 1 \\ \dfrac{\sin\psi_0}{2(n-1)(n+2)} \left[\sin\psi_0 \, P_n(\cos\psi_0) + \cos\psi_0 \, P_n^1(\cos\psi_0) \right] & \text{for } n \geq 2 \end{cases}$$

KERNEL:

REFERENCES:

Gradshteyn-Ryzhik, page 795, equation 7.121.

REMARKS:

Even though the function $P_n^1(\cos\psi_0)$ appears in the spectral coefficient, the spectrum stated above is the spectrum of the zeroth order only.

TRANSFORMATION: DIRICHLET

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint (\cos\psi)^{\gamma}\ f_{IN}(\psi,\alpha)\frac{d\sigma}{4\pi} \qquad \text{where } \gamma > -1$$

SPECTRUM:

$$\lambda_n = \frac{\sqrt{\pi}\ \gamma!}{2^{\gamma+2}\left(\frac{\gamma+n+1}{2}\right)!\left(\frac{\gamma-n}{2}\right)!} = \frac{\gamma!\ \sin\left(\frac{n-\gamma}{2}\right)}{\sqrt{\pi}\ 2^{\gamma+2}}\ \frac{\left(\frac{n-\gamma-2}{2}\right)!}{\left(\frac{n+\gamma+1}{2}\right)!}$$

KERNEL:

REFERENCES:

Robin, Vol. II, page 308, equations 87 and 88.

REMARKS:

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint \operatorname{sgn}(\cos\psi)\, f_{IN}(\psi,\alpha) \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{(-1)^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}} \dfrac{(n-2)!!}{\left(\frac{n+1}{2}\right)!} & \text{for } n \text{ odd} \end{cases}$$

KERNEL:

REFERENCES:

Robin, Vol. II, page 308, equation 92.

REMARKS:

TRANSFORMATION:

Input: $f_{IN}(\psi, \alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint P_N(\cos\psi)\, f_{IN}(\psi,\alpha) \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda_n = \begin{cases} 0 & \text{for } n \neq N \\ \dfrac{1}{2n+1} & \text{for } n = N \end{cases}$$

KERNEL:

$$P_N(\cos\psi) = \sum_{n=0}^{\infty} \frac{\delta_{nN}}{(2n+1)} (2n+1) P_n(\cos\psi)$$

where δ_{nN} is the Kronecker delta

REFERENCES:

REMARKS:

TRANSFORMATION: ADAMS-NEUMANN

Input: $f_{IN}(\psi, \alpha)$

Output: f_{OUT}

EXPLICIT FORM:

$$f_{OUT} = \iint P_p(\cos\psi)\, P_q(\cos\psi)\, f_{IN}(\psi,\alpha) \frac{d\sigma}{4\pi}$$

where p and q are non-negative integers

SPECTRUM:

$$\lambda_n(p,q) = \begin{cases} \dfrac{1}{2s+1} \dfrac{A_{s-p} A_{s-q} A_{s-n}}{A_s} & \text{when } \begin{cases} p+q+n \text{ even} \\ p \le q+n \\ q \le p+n \\ n \le p+q \end{cases} \\ \\ 0 & \text{otherwise} \end{cases}$$

where $s = (p+q+n)/2$ and $A_s = \dfrac{(2s-1)!!}{s!}$

KERNEL:

REFERENCES:

Robin, Vol. II, page 307, equations 85 and 86.

REMARKS:

TRANSFORMATION:

Input: $f_{IN}(\psi,\alpha)$

Output: $\vec{f}_{OUT}$

EXPLICIT FORM:

$$\vec{f}_{OUT} = \iint \frac{2^m \, \Gamma(m+\frac{1}{2}) \sin^m \psi}{\Gamma\left(\frac{1}{2}\right) [1+h^2-2h\cos\psi]^{m+\frac{1}{2}}} \cdot f_{IN}(\psi,\alpha) \begin{Bmatrix} \cos m\alpha \\ \\ \sin m\alpha \end{Bmatrix} \frac{d\sigma}{4\pi}$$

SPECTRUM:

$$\lambda^m_{n+m} = \frac{h^n}{2(n+m)+1}$$

KERNEL:

$$\frac{2^m \Gamma(m+\frac{1}{2}) \sin^m \psi}{\Gamma\left(\frac{1}{2}\right) [1+h^2-2h\cos\psi]^{m+\frac{1}{2}}} \begin{Bmatrix} \cos m\alpha \\ \\ \sin m\alpha \end{Bmatrix} = \sum_{n=0}^{\infty} h^n \, P^m_{n+m}(\cos\psi) \begin{Bmatrix} \cos m\alpha \\ \\ \sin m\alpha \end{Bmatrix}$$

REFERENCES:

Morse-Feshbach, Vol. II, page 1326.

REMARKS:

LIST OF REFERENCES

GEODETIC REFERENCES

- Heiskanen-Moritz, Physical Geodesy, Freeman and Co., 1967.
- Molodenskii, et al., Methods for the Study of the External Gravitational Field and Figure of the Earth, Israel Program for Scientific Translations, 1962. U.S. Department of Commerce Clearinghouse, #TT-61-31207.
- Pick-Picha-Vyskocil, Theory of the Earth's Gravity Field, Elsevier, 1973.

MATHEMATICAL REFERENCES

- Abramowitz-Stegun, Handbook of Mathematical Functions, National Bureau of Standards, 1964; and Dover, 1965 and subsequent.
- Erdelyi (ed.), Higher Transcendental Functions, Vols. I, II, III; McGraw-Hill, 1953.
- Erdelyi (ed.), Tables of Integral Transforms, Vol. I, II; McGraw-Hill, 1954.
- Gradshteyn-Ryzhik, Tables of Integrals, Series, and Products, Academic Press, 1965.
- Magnus-Oberhettinger, Formulas and Theorems for the Functions of Mathematical Physics, Chelsea, 1949.
- Morse-Feshbach, Methods of Theoretical Physics, McGraw-Hill, 1956.
- Robin, Fonctions Sphériques de Legendre et Fonctions Spheroïdales, Vols. I, II, III, Gauthier-Villars, 1957.